NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4090

ANALYSIS OF SHOCK MOTION IN DUCTS DURING DISTURBANCES IN DOWNSTREAM PRESSURE

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Washington
September 1957

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SUMMARY

The effect of small downstream pressure disturbances on the position of a normal shock in a duct with area variation is analyzed. For the analysis, the gas flow is treated as quasi-one-dimensional, and boundary layer is neglected. The analysis shows that there is a first-order lag relation between shock position and small downstream disturbances in pressure which occur at frequencies below a given limit. The time constant and the gain of this lag are expressed in terms of a dimensionless time constant that depends only on the steady-state Mach number of the shock.

INTRODUCTION

When studying the dynamic behavior of propulsion systems for supersonic flight, numerous problems arise that involve the motion of shock waves in ducts. In some of these problems, the shock defines one boundary condition of the transient flow being analyzed. This condition occurs, for example, when the problem is one of determining the dynamics of inlet diffusers for control purposes. In other problems, it is the shock transient itself that is of primary concern. Such is the case when an inlet diffuser with severe buzz characteristics is involved.

Several presentations of the basic theory of shock motion are to be found in the literature; reference 1 is, perhaps, the most comprehensive. In reference 2, however, the discussion is centered on the problems of duct flows. In this reference, a linearized equation governing shock motion is formulated for the case of a normal shock set off from its equilibrium position in an otherwise steady flow. The relaxation time for the return of the shock to equilibrium is thus determined and is used in a discussion of shock-wave stability in diverging or converging ducts.

In the present report, a linearized analysis is made to define the transient imposed on a normal shock by an unsteady downstream flow. The forcing variable of interest is the pressure downstream of the shock. The purpose is to present the dynamic relation between the shock position and downstream pressure in a form readily usable in the study of engine dynamics.

In the analysis, which is made for small perturbations of the shock, the gas flow is treated as quasi-one-dimensional and boundary layer is neglected. The treatment of the unsteady flow downstream of the moving shock is simplified by imposing a frequency limit on the pressure disturbance. The relation sought is derived as a Laplace transfer function, which is considered the most convenient form for most uses.

SYMBOLS

A	cross-sectional area of duct
a	speed of sound
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
$\frac{D}{Dt}$	substantial derivative, $\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}$
k	gain
M	Mach number
P	total pressure
p	static pressure
S	entropy
s	complex operator
t	time
u	gas velocity
v	shock velocity relative to duct
x	position coordinate (positive in downstream direction)
γ	ratio of specific heats

ρ density

σ time constant

σ' dimensionless time constant

ω frequency

Subscripts:

ss steady state

r reference (or datum) values

t total conditions (i.e., stagnation conditions if gas is brought to rest isentropically)

0 entering shock

1 leaving shock

2 at fixed station just downstream of steady-state shock position

Superscript:

* critical conditions (i.e., conditions where gas velocity is equal to sonic velocity)

ANALYSIS

Because of its extreme thinness, a shock wave may be treated as a discontinuity with accuracy. With such a treatment, of course, the unsteady terms vanish from the fundamental relations across the shock. Flow conditions downstream of a moving shock, therefore, can be related to those upstream by steady-state equations, provided that the velocities used are relative to the shock. Hence, for a normal shock, moving at a velocity v with respect to the duct, the pressures on the two sides of it are related as follows:

$$p_1 = p_0 \left[\frac{2\gamma\left(M_0 - \frac{v}{a_0}\right)^2 - (\gamma - 1)}{\gamma + 1} \right] \tag{1}$$

where all velocities are considered positive in the downstream direction.

Consideration is now limited to small perturbations from the steady-state condition. When terms of higher order than these perturbations are neglected, equation (1) yields

$$\Delta p_1 = \frac{4\gamma}{\gamma + 1} p_0 M_0 \left(\Delta M_0 - \frac{\Delta v}{a_0}\right) + \frac{p_1}{p_0} \Delta p_0 \tag{2}$$

where only the Δ quantities, which represent the small perturbations, are time dependent; the coefficients of these variables are steady-state values. The omission of the higher order terms is tantamount to imposing the following conditions:

$$\frac{\Delta v}{M_0 a_0} \ll 1$$

$$\frac{\Delta M_0}{M_0} \ll 1$$

$$\frac{\Delta p_0}{p_0} \ll 1$$

The gas flow upstream of the shock, of course, is not affected by a downstream disturbance. The variables ΔM_0 and Δp_0 in equation (2), therefore, depend only on the area of the duct; and if the area variation is fixed, they are related to the shock position x_0 as follows:

$$\Delta M_0 = \left(\frac{\Delta M_0}{\Delta A_0}\right)_{ss} \left(\frac{\Delta A_0}{\Delta x_0}\right)_{ss} \Delta x_0$$

$$\Delta p_0 = \left(\frac{\Delta p_0}{\Delta A_0}\right)_{ss} \left(\frac{\Delta A_0}{\Delta x_0}\right)_{ss} \Delta x_0$$

The velocity of the shock can also be expressed in terms of shock position since

$$\Delta v = \frac{d(\Delta x_0)}{dt}$$

where the position coordinate x is positive for the downstream direction.

With the use of these expressions for ΔM_0, Δp_0, and Δv, equation (2) becomes

$$\Delta p_1 = \frac{4\gamma}{\gamma + 1} p_0 M_0 \left[\left(\frac{\Delta M_0}{\Delta A_0}\right)_{ss}\left(\frac{\Delta A_0}{\Delta x_0}\right)_{ss} \Delta x_0 - \frac{1}{a_0}\frac{d(\Delta x_0)}{dt}\right] + \frac{p_1}{p_0}\left(\frac{\Delta p_0}{\Delta A_0}\right)_{ss}\left(\frac{\Delta A_0}{\Delta x_0}\right)_{ss} \Delta x_0 \tag{3}$$

The shock movement Δx_0, therefore, is defined in terms of the pressure change Δp_1. This pressure variation occurs at the downstream face of the moving shock. The procedure now is to relate Δp_1 to the disturbance of interest: a pressure disturbance occurring at a station just downstream of the steady-state position of the shock. This fixed station will be referred to as station 2.

The relation between Δp_1 and Δp_2 can be determined from the continuity and momentum equations of unsteady gas flow. For the quasi-one-dimensional flow of a nonviscous gas (with no external forces), the equations are

Continuity:

$$\frac{\partial(\rho u A)}{\partial x} + \frac{\partial(\rho A)}{\partial t} = 0$$

Momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

From these fundamental equations and the relation

$$\frac{p}{\rho^\gamma} = \frac{p_r}{\rho_r^\gamma} e^{(S-S_r)/c_v}$$

the following set of equations can be derived:

$$\left.\begin{aligned}\frac{1}{\rho}\left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right] + a\left[\frac{\partial u}{\partial t} + (u + a)\frac{\partial u}{\partial x}\right] &= \frac{a^2}{c_p}\frac{DS}{Dt} - \frac{a^2}{A}\frac{DA}{Dt}\\ \frac{1}{\rho}\left[\frac{\partial p}{\partial t} + (u - a)\frac{\partial p}{\partial x}\right] - a\left[\frac{\partial u}{\partial t} + (u - a)\frac{\partial u}{\partial x}\right] &= \frac{a^2}{c_p}\frac{DS}{Dt} - \frac{a^2}{A}\frac{DA}{Dt}\end{aligned}\right\} \tag{4}$$

Downstream of the moving shock, of course, each particle of gas maintains a constant value of its entropy (although different particles will have different entropies). The substantial derivative of entropy, therefore, vanishes:

$$\frac{DS}{Dt} = 0$$

In addition, if the area of the duct is not time dependent, the substantial derivative of area reduces as follows:

$$\frac{DA}{Dt} = u \frac{\partial A}{\partial x}$$

With these simplifications, equations (4) become

$$\left.\begin{aligned} \frac{1}{\rho}\left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right] + a\left[\frac{\partial u}{\partial t} + (u + a)\frac{\partial u}{\partial x}\right] &= -\frac{a^2 u}{A}\frac{\partial A}{\partial x} \\ \frac{1}{\rho}\left[\frac{\partial p}{\partial t} + (u - a)\frac{\partial p}{\partial x}\right] - a\left[\frac{\partial u}{\partial t} + (u - a)\frac{\partial u}{\partial x}\right] &= -\frac{a^2 u}{A}\frac{\partial A}{\partial x} \end{aligned}\right\} \tag{5}$$

In linearizing equations (5), it is convenient to make an assumption that appears consistent with the quasi-one-dimensional treatment. It is assumed that partial derivatives with respect to position x for the steady-state condition are of the order of the small-perturbation quantities being considered. That is, the area variation of the duct is assumed to be gradual. Upon linearizing, therefore, for small perturbations from the steady-state condition, equations (5) become

$$\left.\begin{aligned} \frac{1}{\rho}\left[\frac{\partial \Delta p}{\partial t} + (u + a)\frac{\partial \Delta p}{\partial x}\right] + a\left[\frac{\partial \Delta u}{\partial t} + (u + a)\frac{\partial \Delta u}{\partial x}\right] &= -\frac{a^2 u}{A}\frac{\partial \Delta A}{\partial x} \\ \frac{1}{\rho}\left[\frac{\partial \Delta p}{\partial t} + (u - a)\frac{\partial \Delta p}{\partial x}\right] - a\left[\frac{\partial \Delta u}{\partial t} + (u - a)\frac{\partial \Delta u}{\partial x}\right] &= -\frac{a^2 u}{A}\frac{\partial \Delta A}{\partial x} \end{aligned}\right\} \tag{6}$$

where, once again, only the Δ quantities are time dependent.

For the purposes of the present analysis, equations (6) may be greatly simplified. This simplification results from the fact that the bracketed quantities in these equations represent differentiation along the paths of the sonic waves. If the analysis, therefore, is limited to transients that are slow in relation to the transit times for sonic waves, the partial derivatives with respect to time may be omitted from the wave-path derivatives. Such a limitation is not overly restrictive for the problem considered herein. Therefore, with this limit imposed, equations (6) may be written as

$$\left.\begin{aligned}\frac{u+a}{\rho}\frac{\partial\Delta p}{\partial x}+a(u+a)\frac{\partial\Delta u}{\partial x}&=-\frac{a^2u}{A}\frac{\partial\Delta A}{\partial x}\\ \frac{u-a}{\rho}\frac{\partial\Delta p}{\partial x}-a(u-a)\frac{\partial\Delta u}{\partial x}&=-\frac{a^2u}{A}\frac{\partial\Delta A}{\partial x}\end{aligned}\right\}\tag{7}$$

Multiplying the second of equations (7) by $\frac{u+a}{u-a}$ and then adding the result to the first of equations (7) yields the following :

$$\frac{2(u+a)}{\rho}\frac{\partial\Delta p}{\partial x}=-\frac{a^2u}{A}\left(1+\frac{u+a}{u-a}\right)\frac{\partial\Delta A}{\partial x}\tag{8}$$

When equation (8) is multiplied by dx and no distinction is made between the steady-state conditions of stations 1 and 2, it can be integrated to

$$\Delta p_1=\Delta p_2-\frac{\rho_1a_1^2}{A_1}\frac{u_1^2}{u_1^2-a_1^2}\Delta A_1\tag{9}$$

which can be put into the following form:

$$\Delta p_1=\Delta p_2-\frac{\gamma p_1}{A_0}\frac{M_1^2}{M_1^2-1}\left(\frac{\Delta A_0}{\Delta x_0}\right)_{ss}\Delta x_0\tag{10}$$

where the limit can now be defined, frequencywise, as

$$\omega\ll\frac{a_1\pm u_1}{(\Delta x_0)_{max}}$$

The relation between Δp_1 and Δp_2 having thus been determined, the equation relating shock position Δx_0 to the pressure disturbance of interest Δp_2 can now be formulated. This formulation is done by combining equation (10) with equation (3); the manipulation yields

$$\Delta p_2=\left[\frac{4\gamma}{\gamma+1}p_0M_0\left(\frac{\Delta M_0}{\Delta A_0}\right)_{ss}+\frac{p_1}{p_0}\left(\frac{\Delta p_0}{\Delta A_0}\right)_{ss}+\frac{\gamma p_1}{A_0}\frac{M_1^2}{M_1^2-1}\right]\left(\frac{\Delta A_0}{\Delta x_0}\right)_{ss}\Delta x_0-\frac{4\gamma}{\gamma+1}\frac{p_0M_0}{a_0}\frac{d(\Delta x_0)}{dt}\tag{11}$$

From equation (11) it can readily be verified that, in the Laplace domain, the first-order lag relation

$$\frac{\Delta x_0(s)}{\Delta p_2(s)} = k\,\frac{1}{1 + \sigma s} \tag{12a}$$

exists. The time constant σ and the gain k are defined in terms of a dimensionless quantity σ' as follows:

$$\sigma = \frac{1}{a_{t_0}}\,\frac{A_0}{\left(\dfrac{\Delta A_0}{\Delta x_0}\right)_{ss}}\,\sigma' \tag{12b}$$

$$k = -\,\frac{\gamma + 1}{4\gamma}\,\frac{1}{p_0}\,\frac{1}{M_0}\,\frac{a_0}{a_{t_0}}\,\frac{A_0}{\left(\dfrac{\Delta A_0}{\Delta x_0}\right)_{ss}}\,\sigma' \tag{12c}$$

where

$$\sigma' = \frac{-M_0\,\dfrac{a_{t_0}}{a_0}\,\dfrac{A_0^*}{A_0}}{M_0\left(\dfrac{\Delta M_0}{\Delta\dfrac{A_0}{A_0^*}}\right)_{ss} + \dfrac{\gamma + 1}{4\gamma}\,\dfrac{p_1}{p_0}\,\dfrac{P_0}{p_0}\left(\dfrac{\Delta\dfrac{p_0}{P_0}}{\Delta\dfrac{A_0}{A_0^*}}\right)_{ss} + \dfrac{\gamma + 1}{4}\,\dfrac{p_1}{p_0}\,\dfrac{A_0^*}{A_0}\,\dfrac{M_1^2}{M_1^2 - 1}} \tag{12d}$$

Equation (12d) can be reduced to a simpler form by using the familiar functions of M_0 that define the parameters a_{t_0}/a_0, A_0^*/A_0, p_1/p_0, P_0/p_0, and M_1 (ref. 3) and by considering the quantities $\left(\dfrac{\Delta M_0}{\Delta\dfrac{A_0}{A_0^*}}\right)_{ss}$ and $\left(\dfrac{\Delta\dfrac{p_0}{P_0}}{\Delta\dfrac{A_0}{A_0^*}}\right)_{ss}$ to be equal to the corresponding derivatives. The reduced form is as follows:

$$\sigma' = \frac{\dfrac{2(\gamma + 1)}{\gamma - 1}\,M_0\left(1 + \dfrac{\gamma - 1}{2}\,M_0^2\right)^{1/2}}{1 + \dfrac{\gamma^2 + 1}{\gamma - 1}\,M_0^2} \tag{12e}$$

Thus, the quantity σ', which is called the dimensionless shock time constant, is dependent only on the steady-state value of the shock Mach number M_0. A curve showing the variation of the dimensionless time constant with shock Mach number is presented in figure 1. (In determining this curve, a value of 7/5 was used for γ.) As can be seen from the curve, σ' varies only slightly for a wide range of Mach numbers; it is, therefore, a convenient parameter to use.

With the curve of figure 1, it becomes a simple procedure to calculate the time constant σ and the gain k that define the shock dynamics for a given duct and initial condition. As can be seen from equation (12b), the value of σ will be positive for a diverging duct $\left(\frac{\Delta A_0}{\Delta x_0} > 0\right)$ and negative for a converging duct $\left(\frac{\Delta A_0}{\Delta x_0} < 0\right)$. These facts, of course, are consistent with observed stability phenomena of normal shocks and with the conclusions of reference 2.

For the special case of a constant-area duct $\left(\frac{\Delta A_0}{\Delta x_0} = 0\right)$, equations (12b) and (12c) show that both the shock time constant σ and the gain k are infinite. For this case, therefore, a more useful form of equation (12a) is as follows:

$$\frac{\Delta x_0(s)}{\Delta p_2(s)} = \frac{1}{\frac{1}{k} + \frac{\sigma}{k} s}$$

which, for infinite σ and k, gives

$$\frac{\Delta x_0(s)}{\Delta p_2(s)} = \frac{1}{-\left(\frac{4\gamma}{\gamma + 1} \frac{p_0 M_0}{a_0}\right)s}$$

In a constant-area duct, therefore, an integrator action exists between shock position and a downstream disturbance in pressure, the integrator rate being

$$\frac{1}{\frac{4\gamma}{\gamma + 1} \frac{p_0 M_0}{a_0}}$$

In other words, the shock wave continues to move as long as the disturbance is present. This, again, is consistent with experimental observations.

CONCLUDING REMARKS

According to the small-perturbation analysis, a downstream disturbance in pressure (within a specified frequency limit) causes the position of a normal shock in a duct to change in first-order lag manner. The time constant and gain of this lag are functions of steady-state conditions upstream of the shock, the configuration of the duct, and a dimensionless time constant that depends only on the steady-state value of the shock Mach number. For the special case of a constant-area duct, this lag reduces to an integrator action.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, July 12, 1957

REFERENCES

1. Courant, R., and Friedrichs, K. O.: Supersonic Flow and Shock Waves. Pure and Applied Mathematics. Vol. I. Interscience Pub., Inc., 1948.

2. Kantrowitz, Arthur: The Formation and Stability of Normal Shock Waves in Channel Flows. NACA TN 1225, 1947.

3. Ames Research Staff: Equations, Tables, and Charts for Compressible Flow. NACA Rep. 1135. (Supersedes NACA TN 1428.)

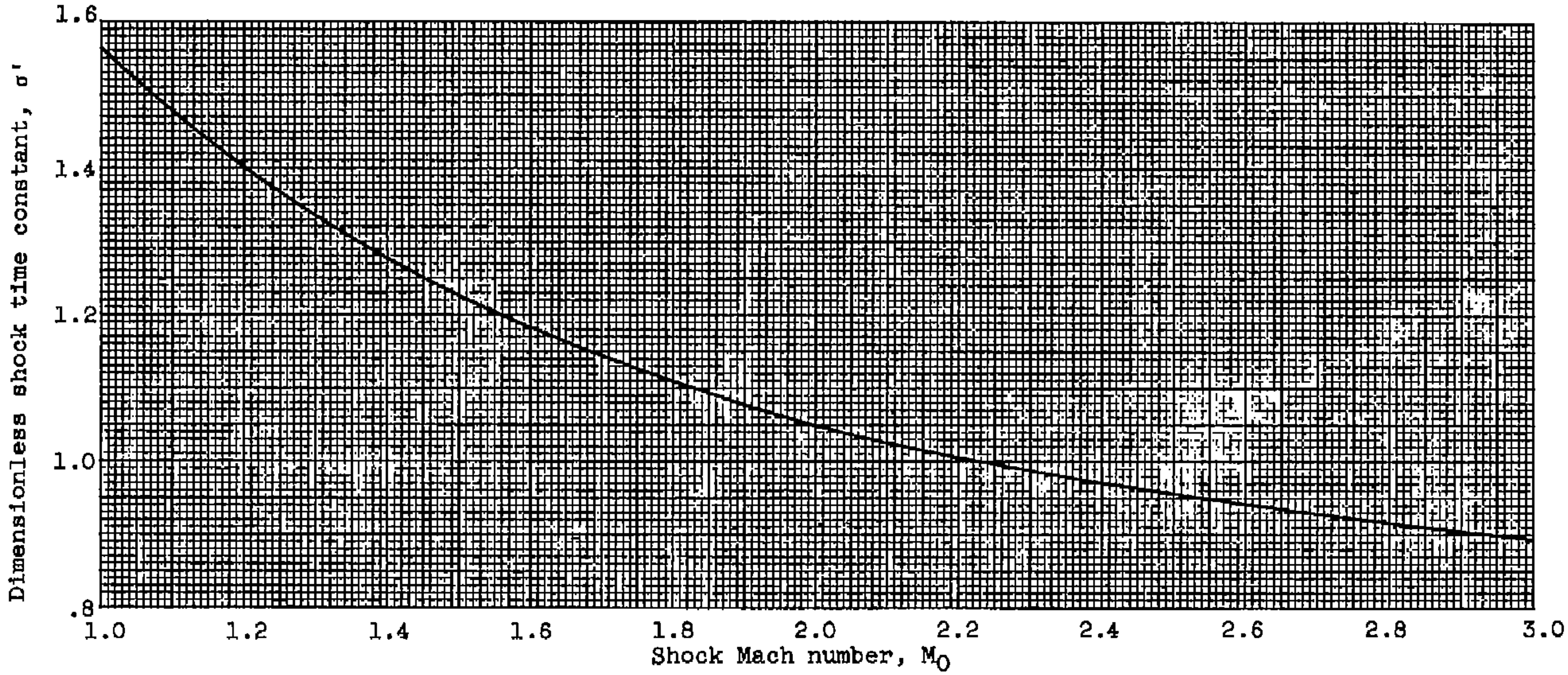

Figure 1. - Variation of dimensionless shock time constant with shock Mach number. (Ratio of specific heats, 7/5.)

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